

AZUMAYA OBJECTS IN TRIANGULATED BICATEGORIES

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ABSTRACT. We introduce the notion of Azumaya object in general homotopy-theoretic settings. We give a self-contained account of Azumaya objects and Brauer groups in bicategorical contexts, generalizing the Brauer group of a commutative ring. We go on to describe triangulated bicategories and prove a characterization theorem for Azumaya objects therein. This theory applies to give a homotopical Brauer group for derived categories of rings and ring spectra. We show that the homotopical Brauer group of an Eilenberg-Mac Lane spectrum is isomorphic to the homotopical Brauer group of its underlying commutative ring. We also discuss tilting theory as an application of invertibility in triangulated bicategories.

1. INTRODUCTION

The notion of Azumaya algebra over a commutative ring k generalizes the notion of a central simple algebra over a field. This was introduced by Azumaya [Azu51] in the case that k is local and by Auslander-Goldman [AG60] for general k . The main classification theorem of Azumaya algebras says that the set of Morita-equivalence-classes of Azumaya k -algebras is the maximal subgroup of the monoid formed by Morita-equivalence-classes of k -algebras under \otimes_k . This is the Brauer group of k , an invariant which carries interesting algebraic and geometric information.

1.1. Foundations of homotopical Brauer theory. This work aims to develop foundations for Brauer theory in homotopical settings. We consider Azumaya objects in closed autonomous monoidal bicategories, and in particular focus on the triangulated bicategories arising as homotopy bicategories of rings and ring spectra. (The term “closed” refers to the existence of internal homs, and “autonomous” refers to opposites such as opposite algebras. See Sections 3.2 and 3.3.)

For the presentation here, we have three audiences in mind: The algebraic audience, for whom Azumaya algebras and Brauer groups of commutative rings are quite familiar; the categorical audience, for whom bicategories and invertibility therein are quite familiar; and the topological audience for whom structured ring spectra and the homotopical algebra thereof are quite familiar. We expect few readers to be in the intersection of these three audiences, and thus have attempted to write for the union of their complements. Each particular reader is encouraged to focus on sections of material with which they are less familiar, remaining patient with the sections written for other readers.

In Sections 1 and 2 we introduce and motivate our results in the discrete and homotopical cases with a minimum of bicategorical language. In Section 2.1 we review classical Brauer groups of commutative rings as a special case of our general approach. A survey of the relevant bicategorical background is then presented in Section 3, with the general characterization of Azumaya objects in Section 3.5. We develop the notion of Azumaya object for triangulated bicategories in Section 4, and discuss homotopical (derived) applications in Section 5.

Now we state the main definitions and results.

Definition 1.1 (Eilenberg-Watts Equivalence). Let A and B be 0-cells of a bicategory \mathcal{B} . We say A is Eilenberg-Watts equivalent to B if and only if there exists an invertible 1-cell $T : A \rightarrow B$.

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Definition 1.2 (Brauer Group, Azumaya Objects). Let \mathcal{B} be a closed autonomous monoidal bicategory with unit 0-cell k . The *Brauer group* of \mathcal{B} , denoted $Br(\mathcal{B})$, is the group of 1-cell-equivalence-classes (Eilenberg-Watts equivalence classes) of 0-cells A for which there exists a 0-cell B such that $A \otimes_k B$ is Eilenberg-Watts equivalent to k . Such 0-cells are called *Azumaya objects*.

These definitions can be compelling from a categorical or conceptual point of view, but it is unsatisfying to work with Azumaya objects defined in this way. Our characterization theorems of Azumaya objects generalize the classical characterization theorem, making computational work with these objects possible. For example, we have the following:

Theorem (See 3.14 iii). *Let k be a commutative ring spectrum, and A a k -algebra which is cofibrant as a k -module. Then A is Azumaya over k if and only if A is faithfully projective (Definition 2.5) over k and the canonical map*

$$A \wedge_k A^{\text{op}} \rightarrow F_k(A, A)$$

is a weak equivalence.

Using the general characterizations of Azumaya objects, we show in Section 5 that the homotopical Brauer groups defined here agree with those of [Toë10] for derived categories of rings and of [BRS10] for commutative S -algebras.

Remark 1.3. What we have termed “Eilenberg-Watts equivalence” is sometimes called “Morita equivalence”, or sometimes “standard Morita equivalence” [KZ98]. The term “Morita equivalence” is also used frequently for monoid objects to mean that their categories of modules are equivalent. The fundamental theorem of Morita theory states that these notions coincide for algebras over a commutative ring and their module categories. Moreover, Rickard [Ric89] has shown that these notions coincide for algebras over a commutative ring and their derived categories of modules. However it is known that this coincidence does not generalize to derived categories of differential graded algebras or ring spectra [Shi06]. Work of Dugger and Dugger–Shipley analyzes the situation for differential graded algebras and ring spectra; see [DS07b, DS07a] and their related articles. The survey [Joh08] compares the two notions abstractly for closed bicategories.

We have elected to introduce new terminology in hopes of clarifying the distinction between the two notions. See [Hov09] for a discussion of Eilenberg-Watts theorems in various settings.

The classical treatment of Azumaya algebras depends heavily on localization arguments and reducing to the case of algebras over fields; one feature of this work is that our treatment is independent of ideal theory. By introducing a bicategorical context we are able to generalize and unify the basic theory of Azumaya algebras and Brauer groups, giving elementary formal proofs of the main classification theorems. Similar ideas have appeared in algebraic and categorical literature—notably [Par75, Vit96, VOZ98, BV02]. We extend this approach to homotopical settings, giving a classification of Azumaya objects via invertibility in triangulated bicategories in Section 4.2.

Notation 1.4. Throughout, k will denote the unit of a symmetric monoidal category \mathcal{C}_k . In our applications, k will often be a commutative d.g. algebra or ring spectrum. We let \mathcal{M}_k be the bicategory of k -algebras and bimodules.

When \mathcal{C}_k has a suitable model structure (described in Section 4.1, following [SS00]), then there is an induced model structure on $\mathcal{M}_k(A, B)$ for k -algebras A and B . We let $\mathcal{D}_k(A, B)$ denote the corresponding homotopy category. Taking as 0-cells the collection of k -algebras which are cofibrant as k -modules, we give one set of conditions in Proposition 4.13 which guarantee that \mathcal{D}_k forms a triangulated bicategory with the additional closed autonomous monoidal structure necessary for the homotopical Brauer theory of Section 3.5.

1.2. Brauer groups of Eilenberg-Mac Lane spectra. One application of the foundations developed here is an immediate computation of the Brauer group of an Eilenberg-Mac Lane spectrum. We include the proof because it is short and because it illustrates the utility of the setting we develop: that which should be formal (because the homotopical algebra of Eilenberg-Mac Lane spectra is equivalent to the ordinary algebra of their underlying rings) has become straightforward (because the Eilenberg-Mac Lane functor induces a local equivalence of triangulated bicategories).

Theorem 1.5. *Let k be a commutative ring. The Eilenberg-Mac Lane functor to symmetric spectra*

$$H: \mathcal{C}_k \rightarrow \mathcal{C}_{Hk}$$

induces an isomorphism of Brauer groups

$$H: Br(\mathcal{D}_k) \rightarrow Br(\mathcal{D}_{Hk}).$$

Proof. By Proposition 4.14 we have closed autonomous monoidal triangulated bicategories \mathcal{D}_k and \mathcal{D}_{Hk} . The Eilenberg-Mac Lane functor H is a weak monoidal Quillen equivalence, which means that H is a lax monoidal functor (but its adjoint is not necessarily monoidal). Thus H sends differential graded k -algebras to Hk -algebras, and sends (A, B) -bimodules to (HA, HB) -bimodules [Shi07]. We have natural equivalences in $\mathcal{D}_{Hk}(HC, HB)$

$$H(X \wedge_A W) \simeq HX \wedge_{HA} HW$$

for $X \in \mathcal{D}_k(C, A)$ and $Y \in \mathcal{D}_k(A, B)$. Therefore H induces a pseudofunctor

$$\mathcal{D}_k \rightarrow \mathcal{D}_{Hk}.$$

Moreover, it gives an equivalence

$$\mathcal{D}_k(A, B) \xrightarrow{\sim} \mathcal{D}_{Hk}(HA, HB)$$

for all differential graded k -algebras A and B and hence a bijection on isomorphism classes of invertible bimodules and on Eilenberg-Watts equivalence classes of Azumaya objects. Thus we have an isomorphism of Brauer groups. \square

Remark 1.6. Note that this result depends on the base commutative ring spectrum Hk being Eilenberg-Mac Lane. If Hk is an algebra over another commutative ring spectrum R , then restriction along the unit map $R \rightarrow Hk$ yields a pseudofunctor of bicategories and hence a map of Brauer groups

$$Br(\mathcal{D}_k) \xrightarrow{\cong} Br(\mathcal{D}_{Hk}) \rightarrow Br(\mathcal{D}_R)$$

which may be neither injective nor surjective. The interesting example of [Shi06, §5] shows that restriction from $H\mathbb{Z}$ to the sphere spectrum is non-injective on Eilenberg-Watts equivalence classes of $H\mathbb{Z}$ -algebras.

As an immediate corollary, we have a computation of the Brauer group for an Eilenberg-Mac Lane spectrum of a field in terms of the classical Brauer group:

Corollary 1.7. *If k is a field, then $Br(\mathcal{D}_{Hk}) \cong Br(k)$. In particular, $Br(\mathcal{D}_{Hk}) = 0$ if k is finite or algebraically closed.*

Proof. Proposition 5.3 says that $Br(\mathcal{D}_k) \cong Br(k)$. The result then follows from Theorem 1.5. \square

In the case that k is an algebraically closed field, this was established by [BRS10] using different methods, but also relying on [Toë10] for the comparison with the derived Brauer group of a field.

1.3. Invertibility. Sections 3.4 and 4.2 give a general study of invertibility in triangulated bicategories, and this forms the foundation of the characterization theorems. In two further applications, we recover portions of the derived Morita (Eilenberg-Watts) theory of Rickard [Ric89] and Schwede-Shipley [SS03b]. In Sections 2 and 3 we give additional motivation, examples, and a review of bicategories.

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2. MOTIVATION

To help motivate the generality of Sections 3 and 4, we give parallel introductions of the main classification theorem for classical algebra and for homotopical settings. Both are special cases of the general theory.

2.1. Brauer groups of discrete rings. In this section we review the classical theory of Brauer groups for commutative (discrete) rings. Let k be a commutative ring and let $\mathcal{B} = \mathcal{M}_k$ be the bicategory of k -algebras and their bimodules. A 0-cell A of this bicategory is a k -algebra, and A^e is the enveloping k -algebra. Note that throughout this section we regard A as a right module over A^e and a left module over k . We recall the classical characterization/definition of Azumaya algebras after introducing basic terminology and a well-known lemma:

Definitions 2.1.

- A is called *separable* over k if A is projective as a module over A^e . Since A is always finitely generated over A^e , this is equivalent to the condition that A be a dualizable module over A^e . By the dual basis lemma, this is equivalent to the condition that the coevaluation

$$A \otimes_{A^e} \text{Hom}_{A^e}(A, A^e) \rightarrow \text{Hom}_{A^e}(A, A)$$

be an isomorphism. To motivate this term, it should be noted that when k is a field, A being separable over k implies that A is semi-simple over k and remains semi-simple upon extension of scalars over any field extension of k . When A is also a field, this implies that A is a separable extension of k in the usual sense for fields [Coh03].

- A is called *central* over k if the center of A is precisely k ; this occurs if and only if the unit

$$k \rightarrow \text{Hom}_{A^e}(A, A)$$

is an isomorphism.

- A is called *faithfully projective* over k if both the coevaluation

$$\text{Hom}_k(A, k) \otimes A \rightarrow \text{Hom}_k(A, A)$$

and the evaluation

$$A \otimes_{A^e} \text{Hom}_k(A, k) \rightarrow k$$

are isomorphisms. Note, again by the dual basis lemma, that the coevaluation being an isomorphism is equivalent to A being finitely-generated and projective as a k -module. The evaluation map being an isomorphism implies that $- \otimes_k A$ is *object faithful*, in the sense that $M \otimes_k A = 0$ implies $M = 0$.

Lemma 2.2 (See e.g. [KO74] or [DI71]). *If A is central and separable over k , then the evaluation*

$$\text{Hom}_{A^e}(A, A^e) \otimes_k A \rightarrow A^e$$

is an isomorphism.

Here is the classical theorem, which follows directly from Theorem 3.14. It provides a link between Eilenberg-Watts equivalences and the more calculable conditions of central, separable and faithfully projective.

Theorem 2.3 (See 3.14). *Let k be a commutative ring, and A a k -algebra. The following are equivalent:*

- A is invertible as an (A^e, k) -bimodule and thus A^e is Eilenberg-Watts equivalent to k .
- A is central and separable over k .
- A is faithfully projective over k and the “sandwich map” (unit)

$$A^e \rightarrow \text{Hom}_k(A, A)$$

is an isomorphism.

- There is a k -algebra B such that $A \otimes_k B$ is Eilenberg-Watts equivalent to k .

2.2. Homotopy bicategories. In Section 4.1 we describe a bicategorical structure on the collection of categories $\mathcal{D}_k(A, B)$, where k is a commutative d.g. algebra or ring spectrum, cofibrant as a module over itself, and A and B are cofibrant as k -modules (see Proposition 4.14). This consists mainly of assembling the necessary results from the literature on monoidal model categories, but we have presented a careful treatment of the closed autonomous monoidal structure, which we define in general in Section 3. The bicategories \mathcal{D}_k are our main examples of interest motivating the theory of triangulated bicategories which we develop in Section 4.

We use \wedge to denote the derived tensor product, and F to denote the derived hom. The isomorphisms in the homotopy category are the weak equivalences, and so we switch to this terminology. Let k be a commutative ring, and let A be a differential-graded algebra over k . For the topologically inclined reader, k and A can be taken to be structured ring spectra. Assume moreover that k and A are cofibrant with respect to a given model structure on \mathcal{C}_k .

To begin, we make the following definitions inspired by the classical case. In fact these are symmetric pairs of definitions corresponding to whether we focus on the natural structure of A as an A^e -module or as a k -module.

Definition 2.4 (Central/Sandwiched). We say that A is *central* over k if the natural unit map

$$k \rightarrow F_{A^e}(A, A)$$

is a weak equivalence. Note that $F_{A^e}(A, A)$ is the (topological) Hochschild cohomology of A over k .

We say that A is *sandwiched* over k if the natural unit map (sometimes called the “sandwich map”)

$$A^e \rightarrow F_k(A, A)$$

is a weak equivalence.

Definition 2.5 (Faithfully Projective/Faithfully Separable). We say that A is *faithfully projective* over k if both the coevaluation

$$F_k(A, k) \wedge_k A \rightarrow F_k(A, A)$$

and the evaluation

$$A \wedge_{A^e} F_k(A, k) \rightarrow k$$

are weak equivalences. Note that the first of these two conditions is equivalent to requiring that A be dualizable (in the derived category) over k . (Duality is discussed in Section 3.4.) The second of these two conditions implies a kind of faithfulness over k , as in the classical case.

We say that A is *faithfully separable* over k if both the coevaluation

$$A \wedge_{A^e} F_{A^e}(A, A^e) \rightarrow F_{A^e}(A, A)$$

and the evaluation

$$F_{A^e}(A, A^e) \wedge_k A \rightarrow A^e$$

are weak equivalences. The first of these two conditions is equivalent to requiring that A be dualizable over A^e , and the second is a kind of faithfulness over A^e .

Definition 2.6. We call A an Azumaya object if any of the equivalent conditions below hold.

Theorem 2.7 (See 3.14). *The following statements are equivalent:*

- i. A is an invertible bimodule, and thus $A \wedge_k A^{\text{op}}$ is Eilenberg-Watts equivalent to k .
- ii. A is central and faithfully separable over k .
- iii. A is sandwiched and faithfully projective over k .
- iv. A^e is Eilenberg-Watts equivalent to k .
- v. There exists a k -algebra B such that $A \wedge_k B$ is Eilenberg-Watts equivalent to k .

Remark 2.8. The equivalence of the first three conditions is formal and straightforward. The argument proceeds by explaining that each of the second two conditions is an alternate description of invertibility. It is clear that the first condition implies the last two; one goal of this work is to develop a setting in which the reverse implications are also straightforward.

Another goal of this work is to explain a further characterization of Azumaya objects when (as in the derived case) the ambient bicategory carries a triangulated structure. In this case, the notion of invertibility admits further alternate descriptions in terms of localization. This has been used in the topological case by Baker-Lazarev [BL04] for THH calculations. Here we expand the generality and develop the connection with classical conditions. As with the rest of this theory, the notion of localization has both left (source) and right (target) versions, but we leave these definitions until Section 4 where we will develop more of the underlying bicategorical structure. The following is immediate from Proposition 4.21.

Theorem 2.9. *The following statements are equivalent:*

- i. A is an Azumaya object.
- ii. A is dualizable over A^e and central over k , and A^e is strongly (target-) A -local.
- iii. A is dualizable over k and sandwiched over k , and k is strongly (source-) A -local.

2.3. Invertibility. Proposition 4.21 gives a general characterization of invertible objects in triangulated bicategories. As further applications, we recover results from the derived Morita (Eilenberg-Watts) theory of Rickard [Ric89] and Schwede-Shipley [SS03b]. The object T here is called a *tilting object*.

Proposition 2.10 ([Ric89]). *Let k be a commutative ring, and let \mathcal{D}_k denote the bicategory of k -algebras and derived categories of bimodules. Suppose B is a differential graded k -algebra, and let T be a differential graded right- B -module. Let $A = \text{End}_B(T)$ be the endomorphism algebra of T . If T has the following two properties, then $\mathcal{D}_k(B)$ and $\mathcal{D}_k(A)$ are equivalent as triangulated categories.*

- i. T is a right-dualizable B -module.
- ii. T generates the triangulated category $\mathcal{D}_k(B)$.

Note. Since T is right-dualizable, T is a bounded complex of finitely-generated and projective B -modules. In particular, T is cofibrant and therefore the derived and underived endomorphism algebras are equal.

Proposition 2.11 ([SS03b]). *Let k be a commutative ring spectrum, and let \mathcal{D}_k denote the bicategory of k -algebras and homotopy categories of bimodules. Suppose B is a k -algebra, and let T be a right- B -module. Let $A = F_B(T, T)$ be the endomorphism ring spectrum of T . If T has the following two properties, then $\mathcal{D}_k(B)$ and $\mathcal{D}_k(A)$ are equivalent categories.*

- i. T is right-dualizable as a B -module.
- ii. T generates the triangulated category $\mathcal{D}_k(B)$.

2.4. Additional Examples.

2.4.1. Symmetric monoidal categories. The example of rings and modules motivates a standard construction for a bicategory $\mathcal{M}_{\mathcal{C}}$ from any cocomplete closed symmetric monoidal category, \mathcal{C} . The 0-cells are taken to be the monoids of \mathcal{C} , and 1-cells are taken to be bimodules. Bimodule maps are defined just as in the case of bimodules over rings, and these constitute the 2-cells of $\mathcal{M}_{\mathcal{C}}$. Colimits from \mathcal{C} are used to construct the horizontal composition in $\mathcal{M}_{\mathcal{C}}$, in the same way that coequalizers give the tensor product of two modules over a ring. Limits give $\mathcal{M}_{\mathcal{C}}$ a closed structure in the same way that equalizers give the homomorphisms of two modules over a ring.

Remembering the underlying symmetric monoidal product of \mathcal{C} gives $\mathcal{M}_{\mathcal{C}}$ an autonomous symmetric monoidal structure. Theorem 3.14 thus characterizes the Azumaya monoids of \mathcal{C} , and defines a Brauer group of \mathcal{C} which reduces to the Brauer group of k when \mathcal{C} is the symmetric monoidal category of modules over a commutative ring k .

The Brauer group of a symmetric monoidal category has been defined directly by Pareigis [Par75] and, for a braided monoidal category, by Van Oystaeyen–Zhang [VOZ98]. These definitions (in the symmetric monoidal case) are equivalent to ours, but there the emphasis is on other aspects of the classical theory (e.g. separability and Casimir elements); their work does not give the full characterization of Azumaya algebras presented in Theorem 3.14. Likewise, we do not discuss the variants on Brauer theory described by Pareigis or Van Oystaeyen–Zhang. Vitale [Vit96] gives a

thorough treatment of Brauer groups for symmetric monoidal categories, and a pleasant summary of categorical approaches to this case.

2.4.2. *Ex-spaces or parametrized spectra.* May–Sigurdsson introduce $\mathcal{E}x$, the bicategory of parametrized spectra in [MS06]. The 0-cells of $\mathcal{E}x$ are topological spaces, and the category of 1-cells $\mathcal{E}x(A, B) = \text{Ho}\mathcal{S}_{A \times B}$ is the homotopy category of *parametrized spectra* over $A \times B$. By considering the projection $A \times B \times C \rightarrow A \times C$ and the diagonal $A \times B \times C \rightarrow A \times B \times B \times C$, they describe a natural horizontal composition

$$\odot : \mathcal{E}x(A, B) \times \mathcal{E}x(B, C) \rightarrow \mathcal{E}x(A, C).$$

The bicategory $\mathcal{E}x$ has a closed structure coming from Brown representability, and the external smash product

$$\bar{\wedge} : \text{Ho}\mathcal{S}_A \times \text{Ho}\mathcal{S}_{A'} \rightarrow \text{Ho}\mathcal{S}_{A \times A'}$$

gives $\mathcal{E}x$ a symmetric monoidal structure. The “opposite” of a space A is again A , and the equivalence $\mathcal{E}x(A, B) \simeq \mathcal{E}x(B^{op}, A^{op}) = \mathcal{E}x(B, A)$ is induced by the switch map

$$t : A \times B \rightarrow B \times A.$$

Thus $\mathcal{E}x$ is a closed autonomous monoidal bicategory and the characterization of Azumaya objects in this bicategory follows as in the case of spectra.

2.4.3. *Rings with many objects.* Let \mathcal{A} be a small category enriched in $\mathcal{A}b$, the category of abelian groups. A left \mathcal{A} -module is a functor $\mathcal{A} \rightarrow \mathcal{A}b$, and a right module is a functor $\mathcal{A}^{op} \rightarrow \mathcal{A}b$; morphisms of \mathcal{A} -modules are enriched natural transformations. Now let $\mathcal{R}ingoids$ be the bicategory whose 0-cells are small $\mathcal{A}b$ -categories, 1-cells are bimodules, and 2-cells are natural transformations. The tensor product and hom for bimodules induce horizontal composition with a closed structure, and the tensor product in $\mathcal{A}b$ induces a symmetric monoidal product on $\mathcal{R}ingoids$ making it a closed autonomous monoidal bicategory. The unit object is the one-object category whose endomorphism ring is the ring of integers.

Clearly this construction may be generalized by replacing $\mathcal{A}b$ with a bicomplete closed symmetric monoidal category \mathbf{V} . We denote the bicategory so-constructed by $\mathbf{V}\text{-}\mathcal{R}ingoids$. This is a closed autonomous monoidal bicategory, and thus Theorem 3.14 gives a characterization of Azumaya objects here too.

In the case that $\mathbf{V} = \text{Mod}_k$ is the category of modules over a commutative ring k , there is a natural morphism of bicategories

$$\iota : \mathcal{M}_k \rightarrow \text{Mod}_k\text{-}\mathcal{R}ingoids$$

given by sending a k -algebra A to the corresponding 1-object Mod_k -category, sending an (A, B) -bimodule to the functor whose value on the single object of $\iota(A) \times \iota(B)^{op}$ is the given bimodule, and sending a morphism of bimodules to the induced natural transformation. The unit object is $\iota(k)$, the one-object category with endomorphism ring k . The morphism ι preserves the Azumaya property, and induces a homomorphism of Brauer groups

$$\text{Br}(\mathcal{M}_k) \rightarrow \text{Br}(\text{Mod}_k\text{-}\mathcal{R}ingoids).$$

For each pair of k -algebras, A and B , the functor $\iota : \mathcal{M}_k(A, B) \rightarrow \text{Mod}_k\text{-}\mathcal{R}ingoids(\iota A, \iota B)$ is an equivalence of categories and hence the induced map on Brauer groups is an injection. This map is in fact an isomorphism, as shown by [BV02, 5.7].

2.4.4. *Sheaves.* The Brauer group of a sheaf \mathcal{O} of commutative rings was introduced by Auslander in [Aus66] and by Grothendieck in [Gro68]. By Theorem 3.14 this is the same as the Brauer group of the symmetric monoidal category of \mathcal{O} -modules: a map of sheaves is an isomorphism if and only if it is so locally, and thus an Azumaya object in the bicategory of \mathcal{O} -algebras and their bimodules is precisely the same as a sheaf of Azumaya algebras. This perspective was initiated by Auslander’s work, and has been discussed by Van Oystaeyen–Zhang [VOZ98, 3.5(2); 3.7].

3. CLOSED AUTONOMOUS MONOIDAL BICATEGORIES

Notation 3.1. We use arrows such as $f : M \rightarrow M'$ to denote that f is a 2-cell with source M and target M' , and slashed arrows such as $M : A \rightarrow B$ to denote that M is a 1-cell with source A and target B . We use \circ or juxtaposition to denote vertical composition of 2-cells, and \odot to denote horizontal composition of 1-cells and of 2-cells. We write \odot -composition in “diagrammatic order”, so that a composite of 1-cells

$$A \xrightarrow{M} B \xrightarrow{N} C$$

is denoted $(M \odot N) : A \rightarrow C$.

3.1. Monoidal bicategories. A monoidal bicategory can be defined as a tricategory with one object, in the sense of [GPS95]. In practical terms, this means that the bicategory is equipped with an additional monoidal product on 0-, 1-, and 2-cells, satisfying reasonable associativity and unit constraints. If \mathcal{C}_k is a cocomplete closed monoidal category with unit k , and \mathcal{M}_k the bicategory of monoids and bimodules in \mathcal{C}_k , then \mathcal{M}_k is a symmetric monoidal bicategory with monoidal product induced by that of \mathcal{C}_k .

Some trivial but slightly technical details about the symmetric monoidal structure will be particularly useful:

Lemma 3.2. *Let k be the unit of a symmetric monoidal bicategory, and suppose that $M : C \rightarrow k$ and $N : k \rightarrow D$ are 1-cells. Then there is a natural isomorphism*

$$M \odot N \cong M \otimes N.$$

Remark 3.3. This lemma follows in essentially the same way as does the corresponding fact about multiplications with the same unit. The 1-cell $k : k \rightarrow k$ is the unit for both \odot and \otimes on 1-cells with source or target k . For intuition, it may be helpful to note that in a symmetric monoidal category with product \otimes and unit k , the identity morphism id_k is the unit for both \circ and \otimes on morphisms with source or target k .

Lemma 3.4. *Let $M : C \rightarrow k$ and $N : k \rightarrow D$ be 1-cells of a symmetric monoidal bicategory \mathcal{B} with unit k . Then we have induced 1-cells*

$$D \otimes_k M : D \otimes_k C \rightarrow D, \quad N \otimes_k C : C \rightarrow D \otimes_k C,$$

and the symmetry for \otimes_k gives an isomorphism between two possible composites in $\mathcal{B}(C, D)$:

$$\begin{array}{ccc} C & \xrightarrow{N \otimes C} & D \otimes C \\ M \downarrow & & \downarrow D \otimes M \\ k & \xrightarrow{N} & D \end{array}$$

Proof. In this proof we include subscripts on \otimes and \odot for the reader’s convenience. The isomorphism is given by

$$\begin{aligned} M \odot_k N &= M \otimes_k N \cong N \otimes_k M \\ &\cong (N \otimes_k C) \odot_{D \otimes_k C} (D \otimes_k M). \end{aligned} \quad \square$$

3.2. Autonomous structure. A symmetric monoidal bicategory \mathcal{B} is *autonomous* [DS97] if each 0-cell A has an opposite, A^{op} , providing an involution on \mathcal{B} and such that there are (suitably natural) equivalences of categories

$$\mathcal{B}(A \otimes B, C) \simeq \mathcal{B}(B, A^{op} \otimes C) \quad \text{and} \quad \mathcal{B}(B \otimes A^{op}, C) \simeq \mathcal{B}(B, C \otimes A)$$

for all B and C . As in classical ring theory, where A^{op} is the opposite k -algebra, we write A^e for the enveloping object $A \otimes A^{op}$, and for a 1-cell $M : A \rightarrow B$ we write $M^{op} : B^{op} \rightarrow A^{op}$ for its image under the equivalence

$$\mathcal{B}(A, B) \simeq \mathcal{B}(B^{op}, A^{op}).$$

Often we will drop the term symmetric, and refer to autonomous symmetric monoidal bicategories simply as *autonomous monoidal bicategories*.

3.3. Closed structure. A *closed structure* for a bicategory, \mathcal{B} , defines right adjoints for \odot . For a 1-cell M , the right adjoint to $M \odot -$ is called “right-hom”, or “source-hom”, and denoted $\text{sHom}(M, -)$. The adjoint to $- \odot M$ is called “left-hom”, or “target-hom”, and denoted $\text{tHom}(M, -)$. For $M \in \mathcal{B}(A, B)$ and $W \in \mathcal{B}(A, C)$, we have the 1-cell $\text{sHom}(M, W) \in \mathcal{B}(B, C)$. For $U \in \mathcal{B}(D, B)$, we have the 1-cell $\text{tHom}(M, U) \in \mathcal{B}(D, A)$. The adjunctions are written as

$$\begin{aligned} \mathcal{B}(M \odot V, W) &\cong \mathcal{B}(V, \text{sHom}(M, W)) \\ \mathcal{B}(T \odot M, U) &\cong \mathcal{B}(T, \text{tHom}(M, U)) \end{aligned}$$

The existence of left and right hom functors defines a *closed bicategory*. A thorough description of closed structures can be found in [MS06].

Note that many of the examples in 2 have internal hom functors defining a closed structure. In general, if \mathcal{C}_k is a bicomplete closed symmetric monoidal category, then the bicategory \mathcal{M}_k built from \mathcal{C}_k is a closed autonomous monoidal bicategory.

3.4. Duality and invertibility in bicategories. For general discussion about duality, we consider fixed 1-cells $X : A \rightarrow B$ and $Y : B \rightarrow A$ in a closed bicategory \mathcal{B} .

Definition 3.5 (Dual pair). We say (X, Y) is a dual pair, or “ X is left-dual to Y ” (“ Y is right-dual to X ”), or “ X is right-dualizable” (“ Y is left-dualizable”) to mean that we have 2-cells

$$\eta : A \rightarrow X \odot Y \quad \text{and} \quad \varepsilon : Y \odot X \rightarrow B$$

such that the following composites are the respective identity 2-cells.

$$X \cong A \odot X \xrightarrow{\eta \odot \text{id}} X \odot Y \odot X \xrightarrow{\text{id} \odot \varepsilon} X \odot B \cong X$$

$$Y \cong Y \odot A \xrightarrow{\text{id} \odot \eta} Y \odot X \odot Y \xrightarrow{\varepsilon \odot \text{id}} B \odot Y \cong Y$$

Lemma 3.6. Let k be the unit of a symmetric monoidal category, and A a k -algebra. Let ${}_k A, A_k$ denote the base-change 1-cells induced by the unit map $k \rightarrow A$. Then $({}_k A, A_k)$ is a dual pair with structure maps

$$k \rightarrow {}_k A \otimes_A A_k$$

and

$$A_k \otimes_k {}_k A \rightarrow A.$$

Note. When (X, Y) is a dual pair, we will occasionally refer to this by saying that X is dualizable over B , since the unit condition often amounts to a finiteness of X over B . When (Y, X) is a dual pair, we will say that X is dualizable over A . With this convention, the phrase “dualizable over” always references the target of the evaluation map, and uniquely determines whether we mean left- or right-dualizable.

Definition 3.7 (Invertible pair). A dual pair (X, Y) is called invertible if the maps η and ε are isomorphisms. Equivalently, the adjoint pairs described in Proposition 3.9 are adjoint equivalences.

Duality for monoidal categories has been studied at length; one reference for duality in a bicategorical context is [MS06, §16.4], and there are surely others in the categorical literature. The definition of duality does not require \mathcal{B} to be closed, but we will make use of the following basic facts about duality, some of which do require a closed structure on \mathcal{B} . The following two results can be found in [MS06, §16.4].

Proposition 3.8. A 1-cell $X \in \mathcal{B}(A, B)$ is right-dualizable if and only if the coevaluation

$$\nu : X \odot \text{tHom}(X, B) \rightarrow \text{tHom}(X, X)$$

is an isomorphism. Moreover, this is the case if and only if the map

$$\nu_Z : X \odot \text{tHom}(X, Z) \rightarrow \text{tHom}(X, X \odot Z)$$

is an isomorphism for all 1-cells $Z : B \rightarrow B$.

Proposition 3.9. Let (X, Y) be a dual pair in \mathcal{B} , with $X : A \rightarrow B$ and $Y : B \rightarrow A$.

i. For any 0-cell C , we have two adjoint pairs of functors, with left adjoints written on top:

$$\mathcal{B}(C, A) \xrightleftharpoons[-\odot_Y]{-\odot_X} \mathcal{B}(C, B)$$

$$\mathcal{B}(A, C) \xrightleftharpoons[X \odot -]{Y \odot -} \mathcal{B}(B, C)$$

The structure maps for the dual pair give the triangle identities necessary to show that the displayed functors are adjoint pairs.

- ii. The right dual, Y , is canonically isomorphic to $\mathrm{tHom}(X, B)$, and for any 1-cell $W : D \rightarrow B$, the coevaluation map $W \odot \mathrm{tHom}(X, B) \rightarrow \mathrm{tHom}(X, W)$ is an isomorphism.
- iii. The left dual, X , is canonically isomorphic to $\mathrm{sHom}(Y, B)$, and for any 1-cell $U : A \rightarrow D'$, the coevaluation map $\mathrm{sHom}(Y, A) \odot U \rightarrow \mathrm{sHom}(Y, U)$ is an isomorphism.

Lemma 3.10. *Let $X : A \rightarrow B$ be a 1-cell in $\mathcal{B}(A, B)$. If X is right-dualizable and the unit $A \rightarrow \mathrm{tHom}(X, X)$ is an isomorphism, then the evaluation $X \odot \mathrm{sHom}(X, A) \rightarrow A$ is an isomorphism. Likewise, if X is left-dualizable and the unit $B \rightarrow \mathrm{sHom}(X, X)$ is an isomorphism, then the evaluation $\mathrm{tHom}(X, B) \odot X \rightarrow B$ is an isomorphism.*

Proof. We prove the first statement, leaving the second as an exercise in opposites. Let Y denote the right dual of X . Since X is right-dualizable, Y is left-dualizable and X is isomorphic to the canonical left dual of Y : $X \cong \mathrm{sHom}(Y, B)$. The isomorphism $A \xrightarrow{\cong} \mathrm{tHom}(X, X)$ implies that the unit for the duality is an isomorphism: $A \xrightarrow{\cong} X \odot Y$. Now we have the following commutative square:

$$\begin{array}{ccc} X \odot_B \mathrm{sHom}(X, A) & \xrightarrow{\text{evaluation}} & A \\ \cong \downarrow & & \downarrow \cong \\ \mathrm{sHom}(Y, B) \odot_B \mathrm{sHom}(X, A) & \xrightarrow{\cong} \mathrm{sHom}(Y, \mathrm{sHom}(X, A)) \xrightarrow{\cong} \mathrm{sHom}((X \odot Y), A) & \end{array}$$

where the two vertical isomorphisms are described above, the left-hand isomorphism is a consequence of dualizability for Y , and the right-hand isomorphism is an exercise in adjunction. \square

3.5. Characterization of Azumaya objects. In this section we give a proof of Theorem 3.14 in two stages. Let A be a fixed 0-cell of a closed autonomous symmetric monoidal bicategory \mathcal{B} .

Notation 3.11. Throughout, it will be useful to distinguish between the 0-cell A , the unit 1-cell $A : A \rightarrow A$ and the 1-cell $A : k \rightarrow A^e$ corresponding to $A : A \rightarrow A$ under the autonomy $\mathcal{B}(A, A) \simeq \mathcal{B}(k, A^e)$. We will refer to the latter as the *floor* of A , and denote it as $[A]$, with the motivation that $[A]$ is naturally induced by A , but with its source being the ground ring and its target being the enveloping object, A^e . Thus $[A]$ is a 1-cell with source k and target A^e :

$$[A] : k \rightarrow A^e.$$

We will not make explicit reference to the related 1-cell $A^e \rightarrow k$, but a compatible notation would be the *ceiling*, $\lceil A \rceil$.

As for the 0-cell A and corresponding unit 1-cell, these sit at different levels of the ambient bicategory, and this will be a sufficient distinction for our purposes.

We now give the general characterization of Azumaya objects; Theorem 2.7 follows directly from this. First, we give general analogs of the definitions in Section 2.2:

Definition 3.12 (Central/Sandwiched). We say that A is *central* over k if the natural unit map

$$k \rightarrow \mathrm{tHom}([A], [A])$$

is an isomorphism.

We say that A is *sandwiched* over k if the natural unit map

$$A^e \rightarrow \mathrm{sHom}([A], [A])$$

is an isomorphism.

Definition 3.13 (Faithfully Projective/Faithfully Separable). We say that A is *faithfully projective* over k if both the coevaluation

$$\mathrm{sHom}([A], k) \wedge_k [A] \rightarrow \mathrm{sHom}([A], [A])$$

and the evaluation

$$[A] \wedge_{A^e} \mathrm{sHom}([A], k) \rightarrow k$$

are isomorphisms.

We say that A is *faithfully separable* over k if both the coevaluation

$$[A] \wedge_{A^e} \mathrm{tHom}([A], A^e) \rightarrow \mathrm{tHom}([A], [A])$$

and the evaluation

$$\mathrm{tHom}([A], A^e) \wedge_k [A] \rightarrow A^e$$

are isomorphisms.

Theorem 3.14. *For A as above, the following statements are equivalent.*

- i. $[A]$ is an invertible 1-cell.
- ii. A is central and faithfully separable over k .
- iii. A is sandwiched and faithfully projective over k .
- iv. A^e is Eilenberg-Watts equivalent to k .
- v. There is a 0-cell B such that $A \otimes B$ is Eilenberg-Watts equivalent to k .

Proof. In general, a pair of 1-cells (X, Y) is invertible if and only if X is right-dualizable, Y is isomorphic to the canonical right dual of X , and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *i* and *ii*. Likewise, (Y, X) is invertible if and only if X is left-dualizable, Y is isomorphic to the canonical left dual of X , and both the unit and counit of the duality are isomorphisms. This gives the equivalence of *i* and *iii*. Since (X, Y) is an invertible pair if and only if (Y, X) is such, the first three conditions are seen to be equivalent.

Connecting the last two conditions to the previous ones is essentially a bicategorical diagram chase: Of course if $[A]$ is invertible, then the implication of the last two conditions is clear. For the reverse, suppose that $P : k \rightarrow A \otimes B$ is an invertible 1-cell between k and $A \otimes B$ and let P^* be its inverse. Let $P^e = P \otimes (P^*)^{op}$ and $P^{*e} = P^* \otimes P^{op}$. Then (P^e, P^{*e}) is an invertible pair of 1-cells between the 0-cells k^e and $(A \otimes B)^e \cong A^e \otimes B^e$. Moreover, since (P, P^*) is an invertible pair, $P^e = P \otimes (P^*)^{op} \cong [A \otimes B] \cong [A] \otimes [B]$. The diagram below describes our situation:

Let $Q = P^{*e}$ and let $T = (A^e \otimes [B]) \odot Q$. We have shown that $([A] \otimes [B], Q)$ is an invertible pair of 1-cells, and we will now show that $([A], T)$ is an invertible pair between A^e and k . A straightforward use of associativity shows that $[A] \odot T \cong k$. We now use the symmetry lemma (3.4) for the reverse composite. This shows that $[A]$ is invertible, completing the proof:

$$\begin{aligned} T \odot [A] &= (A^e \otimes [B]) \odot Q \odot [A] \cong (A^e \otimes [B]) \odot ([A] \otimes A^e \otimes B^e) \odot (A^e \otimes Q) \\ &\cong (A^e \otimes [B]) \odot (A^e \otimes [A] \otimes B^e) \odot (A^e \otimes Q) \\ &\cong (A^e \otimes [A] \otimes [B]) \odot (A^e \otimes Q) \\ &\cong (A^e \otimes k) \cong A^e. \end{aligned}$$

□

4. TRIANGULATED BICATEGORIES

We recall first the definitions of localizing subcategory and generator for a triangulated category, and then give a definition (4.4) of triangulated bicategory suitable for our purposes. In particular, under this definition \mathcal{D}_k is a triangulated bicategory.

Definition 4.1 (Localizing subcategory). If \mathcal{T} is a triangulated category with infinite coproducts, a *localizing* subcategory, \mathcal{S} , is a full triangulated subcategory of \mathcal{T} which is closed under coproducts from \mathcal{T} .

Remark 4.2. This is equivalent to the definition for arbitrary triangulated categories of [Hov99], (which requires that a localizing subcategory be thick) because a triangulated subcategory automatically satisfies the 2-out-of-3 property and because in any triangulated category with countable coproducts, idempotents have splittings. See [Nee01, 1.5.2, 1.6.8, and 3.2.7] for details.

Definition 4.3 (Triangulated generator). A set, \mathcal{P} , of objects in \mathcal{T} (triangulated category with infinite coproducts, as above) is a set of *triangulated generators* (or simply *generators*) if the only localizing subcategory containing \mathcal{P} is \mathcal{T} itself.

Definition 4.4 (Triangulated bicategory [MS06, §16.7]). A closed bicategory \mathcal{B} will be called a *triangulated bicategory* if for each pair of 0-cells, A and B , $\mathcal{B}(A, B)$ is a triangulated category with infinite coproducts, and furthermore the local triangulations on \mathcal{B} are compatible as described in the following axioms.

(TC0) The local suspension functors

$$\Sigma_{A,B} : \mathcal{B}(A, B) \xrightarrow{\sim} \mathcal{B}(A, B)$$

assemble as the components of a strong transformation

$$\Sigma : \mathcal{B}(-, -) \rightarrow \mathcal{B}(-, -).$$

This implies, in particular, that for a 1-cell $X : A \rightarrow B$ there are natural structure isomorphisms

$$\Sigma A \odot X \xrightarrow[\cong]{\alpha_X} \Sigma(A \odot X) \cong \Sigma X \cong \Sigma(X \odot B) \xleftarrow[\cong]{\beta_X} X \odot \Sigma B.$$

(TC1) For a 0-cell A , the following composite of structure isomorphisms interchanging suspension coordinates is multiplication by -1:

$$\Sigma^2 A \xrightarrow{\Sigma \beta_A^{-1}} \Sigma(A \odot \Sigma A) \xrightarrow{\alpha_{\Sigma A}^{-1}} \Sigma A \odot \Sigma A \xrightarrow{\beta_{\Sigma A}} \Sigma(\Sigma A \odot A) \xrightarrow{\Sigma \alpha_A} \Sigma^2 A.$$

(TC2) Each of $-\odot -$, $\text{sHom}(-, -)$, and $\text{tHom}(-, -)$ is exact in both variables.

If \mathcal{B} is a triangulated bicategory and P, Q are 1-cells in $\mathcal{B}(A, B)$, we emphasize that \mathcal{B} is triangulated by writing the abelian group of 2-cells $P \rightarrow Q$ as $\mathcal{B}[P, Q]$ and by writing the graded abelian group obtained by taking shifts of Q as $\mathcal{B}[P, Q]_*$. To emphasize the source and target of P and Q , we may also write $\mathcal{B}(A, B)[P, Q]_*$.

Definition 4.5 (\odot -faithful 1-cells). In any locally additive bicategory, \mathcal{B} , a 1-cell $W : A \rightarrow B$ is called *source-faithful* if triviality for any 1-cell $Z : C \rightarrow A$ is detected by triviality of the composite $Z \odot W$. That is, $Z : C \rightarrow A$ is zero if and only if $Z \odot W = 0$. A collection of 1-cells, \mathcal{E} , in $\mathcal{B}(A, B)$ is called *jointly source-faithful* if the objects have this property jointly; that is, $Z = 0$ if and only if $Z \odot W = 0$ for all $W \in \mathcal{E}$. The term *target-faithful* is defined similarly, considering $W \odot -$ instead of $-\odot W$.

Remark 4.6. If \mathcal{C} is an additive monoidal category with monoidal product \odot , the unit object is both source- and target-faithful. In an arbitrary locally additive bicategory \mathcal{B} , if $A \neq B$ then $\mathcal{B}(A, B)$ may not have a single object with this property. In relevant examples, however, the collection of all 1-cells, $\text{ob}\mathcal{B}(A, B)$, does have this property jointly.

Lemma 4.7. *Let \mathcal{B} be a triangulated bicategory, and let $P : A \rightarrow B$ be a generator for $\mathcal{B}(A, B)$. If the collection of all 1-cells, $\mathcal{B}(A, B)$, is jointly source-faithful (resp. target-faithful), then P is source-faithful (resp. target-faithful).*

Proof. Consider the source-faithful case; the target-faithful case is similar. Given any 1-cell

$$Z : C \rightarrow A$$

with $Z \odot P = 0$, let \mathcal{S} be the full subcategory of 1-cells, $W : A \rightarrow B$ for which $Z \odot W = 0$. This is a localizing subcategory of $\mathcal{B}(A, B)$, and by assumption $P \in \mathcal{S}$, so $\mathcal{S} = \mathcal{B}(A, B)$, and hence $Z = 0$. \square

Remark 4.8. Since the functors $P \odot -$ are exact, the property of $P \odot -$ detecting trivial objects is equivalent to $P \odot -$ detecting isomorphisms (meaning that a 2-cell f is an isomorphism if and only if $P \odot f$ is so).

4.1. Homotopy Bicategories. In this section, let \mathcal{C} be a closed symmetric monoidal model category. Let \mathcal{M} be a closed bicategory formed by \mathcal{C} -monoids and their bimodules. We say that \mathcal{M} has a local model structure if each category of 1- and 2-cells $\mathcal{M}(A, B)$ is a model category. We describe further conditions for the model structure on \mathcal{C} which ensure that \mathcal{M} has a local model structure and that the collection of homotopy categories assembles to a closed autonomous monoidal bicategory.

Definition 4.9 (Pushout products). Suppose \mathcal{M} has a local model structure. We say \mathcal{M} has pushout products if, for any 0-cells A, B , and C and for any cofibrations $f : U \rightarrow V$ in $\mathcal{M}(A, C)$ and $g : W \rightarrow X$ in $\mathcal{M}(C, B)$, the pushout product $f \square g$ below is a cofibration in $\mathcal{M}(A, B)$ which is acyclic if either f or g is acyclic.

$$\begin{array}{ccc} U \wedge_C W & \longrightarrow & V \wedge_C W \\ \downarrow & & \downarrow \\ U \wedge_C X & \longrightarrow & \mathbb{P} \end{array} \quad \begin{array}{c} \xrightarrow{f \square g} \\ \searrow \\ V \wedge_C X \end{array}$$

Definition 4.10 (Unit replacement condition). Let A be a 0-cell of \mathcal{M} , and let QA be a cofibrant replacement for the unit 1-cell A in $\mathcal{M}(A, A)$. We say that the unit replacement condition holds for A if, for any cofibrant $X \in \mathcal{M}(A, C)$, the induced map

$$QA \odot X \rightarrow A \odot X \cong X$$

is a weak equivalence in $\mathcal{M}(A, C)$. Note that this condition is independent of the choice of cofibrant replacement QA , and is automatically satisfied if A is cofibrant in $\mathcal{B}(A, A)$.

Pushout products imply that the horizontal composition of 1-cells descends to a homotopy bicategory. The unit replacement condition implies that horizontal composition of 1-cells on the level of homotopy is unital. Next we recall one result which implies that \mathcal{M} has a local model structure, and then give another showing that this local model structure indeed descends to form a homotopy bicategory.

Proposition 4.11 ([SS00, 3.1]). *Let $\mathcal{C} = (\mathcal{C}, \wedge, k)$ be a cofibrantly generated closed monoidal model category which satisfies the monoid axiom and in which each object is small with respect to the whole category. Let $\mathcal{M}_{\mathcal{C}}$ be the bicategory formed by \mathcal{C} -monoids which are cofibrant in \mathcal{C} and their bimodules. Then $\mathcal{M}_{\mathcal{C}}$ has a local model structure, and has pushout products. Moreover, the category of \mathcal{C} -monoids $\text{Mon}(\mathcal{C})$ is a cofibrantly generated model category, and if k is cofibrant then every cofibration in $\text{Mon}(\mathcal{C})$ whose source is cofibrant is also a cofibration upon forgetting to \mathcal{C} .*

Proposition 4.12 (Homotopy bicategory). *Let \mathcal{C} be as above, and assume the following:*

- *The unit, k , is cofibrant in \mathcal{C} .*
- *The unit replacement condition holds for each monoid in \mathcal{C} which is cofibrant in \mathcal{C} .*
- *For each monoid A and cofibrant A -module N , $- \wedge_A N$ takes weak-equivalences of A -modules to weak equivalences in \mathcal{C} .*

Then the collection of homotopy categories $h.\mathcal{M}_{\mathcal{C}}(A, B)$ forms a closed bicategory, which we denote $h.\mathcal{M}_{\mathcal{C}}$.

Proof. We begin with an application of the theory of Quillen functors of two variables [Hov99, §4.2]. Since $\mathcal{M}_{\mathcal{C}}$ has pushout products by Proposition 4.11, each component of horizontal composition is a left Quillen bifunctor

$$\odot : \mathcal{M}(A, C) \times \mathcal{M}(C, B) \rightarrow \mathcal{M}(A, B)$$

and thus induces a bifunctor on $h\mathcal{M}$ which we also denote with \odot . The unit replacement condition ensures that \odot is unital on $\mathcal{M}_{\mathcal{C}}$, and the third condition ensures that $h\mathcal{M}_{\mathcal{C}}(A, B) \simeq h\mathcal{M}_{\mathcal{C}}(A', B')$ if $A \simeq A'$ and $B \simeq B'$ are weak equivalences in $\text{Mon}(\mathcal{C})$ [SS00, Theorem 3.3]. \square

Proposition 4.13. *Let \mathcal{C} be as in Proposition 4.12. Then the derived monoidal product on \mathcal{C} descends to an autonomous monoidal structure on $h\mathcal{M}$.*

Proof. If A and B are cofibrant in \mathcal{C} , then $A \wedge B$ is a model for the derived product, and a cofibrant (A, B) -bimodule is also cofibrant in \mathcal{C} . Moreover, the product in \mathcal{C} preserves bimodule structure and hence induces a bifunctor

$$\wedge : h\mathcal{M}_{\mathcal{C}}(A, B) \times h\mathcal{M}_{\mathcal{C}}(C, D) \rightarrow h\mathcal{M}_{\mathcal{C}}(A \wedge C, B \wedge D)$$

which agrees (for each $A, B, C, D \in \text{Mon}(\mathcal{C})$) with the derived monoidal product in \mathcal{C} . This gives an associative product on the bicategory $h\mathcal{M}_{\mathcal{C}}$, which is unital because k is cofibrant in \mathcal{C} . The autonomous structure on $h\mathcal{M}_{\mathcal{C}}$ is obtained by taking the opposite multiplication on monoids and modules of \mathcal{C} , and the symmetry of the product in \mathcal{C} gives the symmetry in $h\mathcal{M}_{\mathcal{C}}$. \square

We now give our main application, in which k is taken to be a commutative d.g. algebra or commutative ring spectrum and $\mathcal{C} = \mathcal{C}_k$ is taken to be the category of k -modules. One may work with any monoidal model category of spectra, but we must assume that k is cofibrant in \mathcal{C}_k and thus the result applies only to those categories in which the unit is cofibrant (e.g., symmetric spectra or orthogonal spectra). Likewise, if k is a d.g. algebra, one must choose a model structure on \mathcal{C}_k for which the unit is cofibrant (e.g., the injective model structure).

Proposition 4.14. *Let k be a commutative d.g. algebra or commutative ring spectrum, and let \mathcal{C}_k be the symmetric monoidal category of k -modules, with a model structure for which k is cofibrant. Then we have closed autonomous monoidal bicategories $\mathcal{M}_k = \mathcal{M}_{\mathcal{C}_k}$ and $\mathcal{D}_k = h\mathcal{M}_{\mathcal{C}_k}$.*

Proof. The hypotheses of Propositions 4.11 and 4.12 are satisfied by \mathcal{C}_k , so Proposition 4.13 applies: Most of the hypotheses are discussed in [SS00] and so we comment only on the unit replacement condition. Suppose $A \in \text{Mon}_c(\mathcal{C})$, and consider a specific cofibrant replacement for A as an (A, A) -bimodule: the two-sided bar construction $QA = B(A, A, A)$. Then $QA \wedge_A X \simeq B(A, A, X)$, and the latter retracts to $A \wedge_A X$ along the simplicial retraction induced by the unit of A . Thus the induced map

$$QA \wedge_A X \rightarrow A \wedge_A X \simeq X$$

is an equivalence in $\mathcal{D}_k(A, A)$. \square

Remark 4.15. The local model structure of \mathcal{M}_k in the examples above is stable, and thus \mathcal{D}_k is a locally triangulated bicategory. The suspension functor on $\mathcal{D}_k(A, B)$ is given by horizontal composition $\Sigma A \odot -$ and these local triangulations are compatible as described in Definition 4.4, and so \mathcal{D}_k is a triangulated bicategory.

4.2. Invertibility. For this section, we let \mathcal{D} denote a triangulated closed autonomous monoidal bicategory with unit 0-cell k . Bousfield localization for triangulated categories [Kra09, §4.10] generalizes to triangulated bicategories in two ways, using 1-cell composition over either the source or target of a given 1-cell. Recall that we write 1-cell composition in diagrammatic order:

$$\left(C \xrightarrow{M} A \right) \odot \left(A \xrightarrow{T} B \right) = \left(C \xrightarrow{M \odot T} B \right)$$

Definition 4.16. Let $T : A \rightarrow B$ be a 1-cell in $\mathcal{D}(A, B)$.

A 1-cell $M : C \rightarrow A$ is *source- T -acyclic* if $M \odot T = 0$. A 1-cell $N : C \rightarrow A$ is *source- T -local* if

$$\mathcal{D}(C, A)[M, N]_* = 0$$

for all source- T -acyclic 1-cells $M \in \mathcal{D}(C, A)$. The full subcategory of source- T -local 1-cells in $\mathcal{D}(C, A)$ is denoted $\mathcal{D}(C, A)_{\langle \odot T \rangle}$.

A 1-cell $M' : B \rightarrow C$ is *target- T -acyclic* if $T \odot M = 0$. A 1-cell $N' : B \rightarrow C$ is *target- T -local* if

$$\mathcal{D}(B, C)[M', N']_* = 0$$

for all target- T -acyclic 1-cells $M' \in \mathcal{D}(B, C)$. The full subcategory of target- T -local 1-cells in $\mathcal{D}(B, C)$ is denoted $\mathcal{D}(B, C)_{\langle T \odot \rangle}$.

Baker and Lazarev describe the following in the context of spectra, but their methods generalize to our setting. The key observation is that for any 1-cell P whose source is A , $\text{sHom}(T, P)$ is target- T -local. Likewise, if P' is any 1-cell whose target is B , $\text{tHom}(T, P')$ is source- T -local.

Proposition 4.17 ([BL04]). *Let $T : A \rightarrow B$ be a 1-cell in $\mathcal{D}(A, B)$. The adjunctions induced by T factor through the T -local pseudofunctors, which is to say that we have the following diagrams of adjoint transformations:*

$$\begin{array}{ccc} \mathcal{D}(B, -) & \xrightleftharpoons[T \odot -]{\text{sHom}(T, -)} & \mathcal{D}(A, -) \\ & \searrow \text{sHom}(T, -) & \nearrow T \odot - \\ & \mathcal{D}(B, -)_{\langle T \odot \rangle} & \end{array} \quad \begin{array}{ccc} \mathcal{D}(-, A) & \xrightleftharpoons[-\odot T]{\text{tHom}(T, -)} & \mathcal{D}(-, B) \\ & \searrow \text{tHom}(T, -) & \nearrow -\odot T \\ & \mathcal{D}(-, A)_{\langle \odot T \rangle} & \end{array}$$

Proposition 4.18 ([BL04]). *If a 1-cell $T \in \mathcal{D}(A, B)$ is right-dualizable and the unit map induces an isomorphism $A \cong \text{tHom}(T, T)$, then the induced adjoint pair is an equivalence*

$$\mathcal{D}(B, -)_{\langle T \odot \rangle} \simeq \mathcal{D}(A, -).$$

We have a corresponding statement for the case of left-dualizability.

Proof. It follows immediately from Lemma 3.10 that the evaluation map

$$T \odot \text{sHom}(T, M) \rightarrow M$$

is an isomorphism for all M ; this is the counit of the adjunction. Moreover, $\text{sHom}(T, -)$ takes values in the target- T -local category and hence the fact that the unit of the adjunction is an isomorphism follows from the fact that the counit is so. \square

Corollary 4.19. *Let $T : A \rightarrow B$ be as in Proposition 4.18. Then T is target-faithful (Definition 4.5) if and only if localization induces an equivalence between the category of 1-cells $\mathcal{B}(B, B)$ and the target- T -local subcategory. In this case each of the three adjoint pairs of Proposition 4.17 (at left) is an equivalence. We have a corresponding statement for the source-faithful case.*

Proof. If T is target-faithful, then all target- T -acyclics are trivial, and therefore

$$\mathcal{D}(B, -) \simeq \mathcal{D}(B, -)_{\langle T \odot \rangle}.$$

The statement then follows from Proposition 4.18. Conversely, if localization induces an equivalence $\mathcal{D}(B, B) \simeq \mathcal{D}(B, B)_{\langle T \odot \rangle}$ then the proof of Proposition 4.18 shows that T is invertible and hence target-faithful. \square

Definition 4.20. We say that B is *strongly target- T -local* if localization induces an equivalence

$$\mathcal{D}(B, B) \simeq \mathcal{D}(B, B)_{\langle T \odot \rangle}.$$

Likewise, we say that A is *strongly source- T -local* if localization induces an equivalence

$$\mathcal{D}(A, A) \simeq \mathcal{D}(A, A)_{\langle \odot T \rangle}.$$

Combining the previous results yields a characterization of invertible objects in triangulated bicategories with closed autonomous monoidal structure. Applying this to the case $T = [A]$ immediately gives the characterization of Azumaya objects in Theorem 2.9.

Proposition 4.21. *Let $T : A \rightarrow B$ be a 1-cell in \mathcal{D} . The following are equivalent:*

- i. *T is invertible.*
- ii. *a) T is right-dualizable.
b) The unit induces $A \cong \mathrm{tHom}(T, T)$.
c) B is strongly target- T -local.*
- iii. *a) T is left-dualizable.
b) The unit induces $B \cong \mathrm{sHom}(T, T)$.
c) A is strongly source- T -local.*

Remark 4.22. We have the following chain of implications for $T : A \rightarrow B$, and a similar chain for source-faithful and source-local conditions:

$$T \text{ is target-faithful.} \Rightarrow B \text{ is strongly target-}T\text{-local.} \Rightarrow B \text{ is target-}T\text{-local.}$$

Corollary 4.19 shows that the first implication is an equivalence under the conditions of Proposition 4.18, and therefore condition (c) in each of the statements of Proposition 4.21 can be replaced by the corresponding faithfulness condition on T .

4.3. Application to tilting theory. The work in this section allows us to give a unified proof of results from the tilting theory of [Ric89] and [SS03b]:

Proof of 2.10 and 2.11. Let \tilde{T} denote T regarded as a bimodule over $A = \mathrm{tHom}(T, T)$. Since T is right-dualizable, \tilde{T} is right-dualizable in $\mathcal{D}_k(A, B)$. Moreover, $A \simeq \mathrm{tHom}(\tilde{T}, \tilde{T})$.

By Proposition 4.14, \mathcal{D}_k is a triangulated closed autonomous monoidal bicategory. Since k is the unit of \mathcal{D}_k , the 1-cells of $\mathcal{D}_k(k, B)$ are jointly target-faithful (Definition 4.5). Since T generates $\mathcal{D}_k(k, B)$, Lemma 4.7 shows that T is target-faithful and this means that \tilde{T} is also target-faithful. The result then follows from Corollary 4.19 and Proposition 4.21. \square

5. HOMOTOPICAL BRAUER GROUPS

This section describes homotopical Brauer groups. We also give explicit comparisons between the Brauer group as characterized by Theorems 2.9 and 3.14 and as it appears in related work on Brauer groups in homotopical settings. We begin with the derived Brauer group of a ring and then address the Brauer groups of ring spectra.

5.1. The derived Brauer group of a ring.

Definition 5.1. We refer to $Br(\mathcal{D}_k)$ as the *derived Brauer group* of k , to distinguish it from the generally different classical Brauer group, $Br(k)$.

Toën [Toë10] introduces the notion of *derived Azumaya algebras* (the two conditions appearing in Proposition 5.2), and describes a Brauer group formed by Eilenberg-Watts equivalence classes of such. We show that the notion of derived Azumaya algebra is equivalent to the notion of Azumaya object in the derived category, and therefore the resulting Brauer groups are isomorphic.

Proposition 5.2. *Let k be a graded commutative ring and let A be a k -algebra, regarded as a left-module over k and a right module over the enveloping algebra A^e . Then A is an Azumaya object of \mathcal{D}_k if and only if the following two conditions hold:*

- i. *The underlying k -module of A is a compact generator of the triangulated category $\mathcal{D}_k(k, k)$.*
- ii. *The map $\mu : A^e \rightarrow F_k(A, A)$ is an equivalence.*

Proof. Toën [Toë10, 1.8] shows that the conditions above are equivalent to the condition that A^e be Eilenberg-Watts equivalent to k in the bicategory \mathcal{D}_k , and this is one of the conditions appearing in the characterization Theorem 3.14. \square

Thus we have the following:

Proposition 5.3 ([Toë10, 1.12]). *Let k be a field. Then $Br(\mathcal{D}_k)$ is isomorphic to the classical Brauer group $Br(k)$.*

For k not a field, this result does not generalize, although there is of course always a homomorphism

$$Br(k) \rightarrow Br(\mathcal{D}_k).$$

5.2. The Brauer group of a ring spectrum. When k is a commutative ring spectrum Baker, Richter and Szymik have introduced and studied the notion of topological Azumaya k -algebra [BRS10]. Their definition (the three conditions of Proposition 5.4 below) makes sense in any modern monoidal model category of spectra, and we show it is equivalent to the definition of Azumaya used here. Since the various modern categories of spectra are all strong monoidal Quillen equivalent [SS03a], the results of homotopical Brauer theory transfer between any of them.

Recall that $[A] : k \rightarrow A^e$ denotes A regarded as a right-module over A^e , and ${}_k A_k = [A]_k$ is the underlying (k, k) -module of A .

Proposition 5.4. *Let k be a commutative ring spectrum such that \mathcal{D}_k has the structure of a closed autonomous monoidal bicategory. Let A be a k -algebra, regarded as a left-module over k and a right module over the enveloping algebra A^e . Then A is an Azumaya object in \mathcal{D}_k if and only if the following conditions hold:*

- i. *A is dualizable in the homotopy category of k -modules.*
- ii. *The map $A^e \rightarrow F_k(A, A)$ is a weak equivalence of bimodules over A^e .*
- iii. *A is faithful as a k -module.*

Proof. Baker, Richter, and Szymik show that these conditions imply that $[A]$ is central and separable over k [BRS10, 1.3, 1.4]. They are working with S -algebras, but those results apply equally well in any modern category of spectra. Now if $[A]_k = [A] \odot A_k$ is faithful, then $[A]$ must be target-faithful (Definition 4.5). Remark 4.22 explains that under these circumstances target-faithfulness of $[A]$ is equivalent to A^e being strongly target- $[A]$ -local, and thus A is Azumaya by Proposition 4.21.

For the converse, we again refer to Proposition 4.21. Lemma 3.6 shows that $[A]$ being left dualizable implies $[A]_k$ is left dualizable, and under these circumstances k being strongly source- $[A]$ -local is equivalent to $[A]$ being source-faithful (Remark 4.22). Since restriction of scalars along the unit map $k \rightarrow A^e$ is faithful, the composite $[A]_k$ must also be faithful. \square

One minor subtlety remains in the choice of equivalence relation with which we form a Brauer group. There are two standard choices, Eilenberg-Watts equivalence and Brauer equivalence. We finish by showing that the two resulting groups are isomorphic. This is a classical fact for discrete rings which holds for formal reasons.

Definition 5.5 (Brauer equivalence). Two k -algebras A_1 and A_2 are called (topological) Brauer equivalent if there are faithful, dualizable, cofibrant k -modules M_1 and M_2 such that

$$A_1 \wedge_k F_k(M_1, M_1) \simeq A_2 \wedge_k F_k(M_2, M_2)$$

as k -algebras.

Proposition 5.6 ([BRS10, 2.4]). *The collection of Azumaya k -algebras modulo Brauer equivalence forms a group.*

Lemma 5.7. *The group of Brauer equivalence classes of Azumaya k -algebras is isomorphic to the group of Eilenberg-Watts equivalence classes of such algebras.*

Proof. Brauer equivalence implies Eilenberg-Watts equivalence and thus there is a surjective group homomorphism from Azumaya algebras modulo Brauer equivalence to Azumaya algebras modulo Eilenberg-Watts equivalence. But if A is Eilenberg-Watts equivalent to the ground ring, k , then there is an invertible bimodule M giving the equivalence, and $A \simeq F_k(M, M)$, so A is also Brauer equivalent to k . Thus this homomorphism is an isomorphism. \square

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